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Percolation under rotational constraint: a finite-size scaling study

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Abstract. A percolation model, spiral percolation, in which a rotational constraint is operative is studied by the finite-size scaling method. The critical percolation probability p_c and the critical exponents ν, β, γ, τ, σ and also the fractal dimension D of the spanning cluster are determined. Evidence is obtained for a scaling form of the cluster distribution function.

1. Introduction

The effect of external constraints on the percolation process has been mainly studied through a model of directed percolation (Kinzel 1983). In directed percolation a global bias constrains the flow in particular directions, say, only upwards and towards the right. The percolation cluster grows in the preferred directions leading to anisotropic scaling and direction-dependent critical behaviour. Recently, a new type of percolation process, spiral percolation, has been suggested (Ray and Bose 1988). In this model, the spiral percolation paths are defined on the spanning cluster generated in undirected percolation. Each step of a percolation path proceeds either straight or in a specific rotational direction, say clockwise. Spiral percolation occurs if a cluster obeying the rotational constraint mentioned spans the underlying undirected cluster. Monte Carlo simulation, assuming a finite-size scaling hypothesis, has been performed on the square lattice for lattice size up to 60×60 to determine the spiral percolation threshold p_c and the correlation length exponent ν_s. The model as defined describes spiral percolation on an undirected spanning cluster embedded in a square lattice. In this paper, we study spiral percolation directly on the square lattice, the percolation paths obeying the rotational constraint as in the case of the previous model. We present the results of a finite-size scaling study of spiral percolation in the critical region for lattice size up to 140×140. In section 2 we describe the finite-size scaling theory of percolation and the procedure followed by us for calculating the various critical quantities. Section 3 contains a description of the results obtained. Section 4 gives a discussion of these results.

2. Finite-size scaling method

We consider spiral site percolation on a square lattice of size $L \times L$. The rotational constraint allows the percolation cluster to grow either in the forward or in the clockwise

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direction. The spiral percolation threshold $p_c(L)$ is determined by the binary search method (Hoshen and Kopelman 1976). We start from a central site of the lattice which is called the origin. Initially, from the origin, one can proceed in any one of the four possible directions. The nearest-neighbour (NN) sites in these directions are occupied with probability p by using a random number generator. With each occupied site a variable JVISIT(IV, IOCC) is associated, IOCC is the site index and I_v the direction index. The sites of the lattice are numbered according to a particular sequence; the site index is the number associated with a site. A site can be reached from four directions south, west, north and east. The corresponding direction indices are 1, 2, 3 and 4 (figure 1(a)). The rotational constraint implies the following: if a site has direction index 1, then occupy, with probability p, only those of the NNs which are towards the north and east of the site.

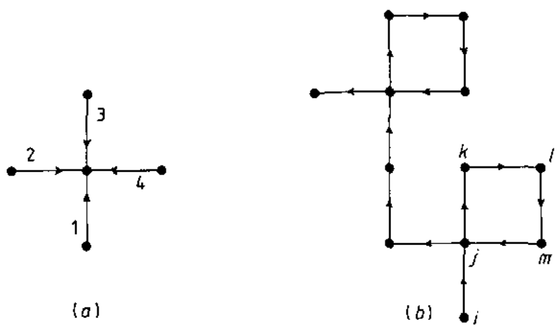

Figure 1. (a) Direction indices of a site corresponding to the four different directions from which the site can be reached. (b) An example of a spiral site cluster on the square lattice. The arrows on the bonds indicate the allowed spiral directions of flow from site i.

Initially, the JVISIT variable is assigned the value zero for all sites. As soon as a site is occupied, the corresponding JVISIT variable is given the value IV where IV is the direction index corresponding to the direction from which the site is occupied. All growing sites are put on a list and the walk-search is carried out for each. The growth of a cluster of occupied sites stops only when all the perimeter sites are unavailable for occupation. Figure 1(b) shows a typical cluster grown obeying the rotational constraint. Due to the nature of the constraint, loops are an essential feature of the growing spirally connected cluster. In our computer algorithm, the possibility of loop formation while growing a cluster is taken care of in the following manner: we refer to figure 1(b). The site i is the origin. In the first step when j is occupied, JVISIT$(1, j) = 1$. Next time j is approached (from site m), the computer algorithm checks whether JVISIT$(4, j)$ is zero. If it is non-zero then the site j has been approached from the east previously and so cannot be occupied again. For the particular cluster in figure 1(b), since JVISIT$(4, j) = 0$, the site j is reoccupied and JVISIT$(4, j)$ assigned the value 4. Thus loop formation and continuation of cluster growth are possible.

Following the binary search method we start growing a cluster with a particular value of p, say p_0. If the cluster grown spans (does not span) the lattice in either the east–west or north–south direction, p_0 is decreased (increased) by a small amount. The same random number sequence is then used to get estimates $p_1(L)$ and $p_2(L)$ which bound an interval containing the true threshold value $p(L)$. By successive binary chopping of this interval one can determine $p_c(L)$ with a specified accuracy. The whole process is then repeated N times ($N \times L \times L$ is of the order of 10^6–10^7) using different random number sequences. The average value $\langle p_c(L)\rangle$ of all the estimates obtained is taken as an estimate for the percolation threshold. The spread in the estimates is related to the correlation length exponent ν (correlation length $\xi \sim |p-p_c|^{-\nu}$ as $p \to p_c$) through the finite-size scaling formula (Levinshtein *et al* 1976, Reynolds *et al* 1980):

$$\Delta(L) = [\langle p_c^2(L)\rangle - \langle p_c(L)\rangle^2]^{1/2} \sim L^{-1/\nu}. \tag{1}$$

$p_c(\infty)$, the percolation threshold in the limit of an infinitely large lattice is obtained through the finite-size scaling formula

$$|p_c(\infty) - \langle p_c(L)\rangle| \sim L^{-1/\nu}. \tag{2}$$

Equations (1) and (2) determine the exponent ν and $p_c(\infty)$.

To determine other critical exponents we proceed as follows. For a particular lattice size $L \times L$, we choose p to be equal to $\langle p_c(L)\rangle$. Ten thousand clusters are grown for this value of p and the distribution of clusters of various sizes determined. Clusters of neighbouring sizes are counted in one bin. For example the ith bin contains clusters of sizes in the range 2^{i-1}–(2^i-1). Let $n_s(p)$ be the number of clusters of size s per site of the lattice. For an infinitely large lattice p_c has a sharp unique value and in the critical region $p \to p_c$, one can define the following critical quantities:

$$\text{average cluster size } \chi \sim \sum_s s^2 n_s \sim |p-p_c|^{-\gamma}. \tag{3}$$

Probability P that a site belongs to the infinite (spanning) cluster goes as

$$P \sim (p-p_c)^{\beta}. \tag{4}$$

In a finite system, any one of the quantities defined above depends not only on p but also on the linear dimension L of the lattice. True critical behaviour occurs only in the limit of infinitely large lattices but an estimate of the critical exponents, e.g. β and γ, can be obtained from the studies of finite systems by assuming a finite-size scaling hypothesis (Stauffer 1985) which leads to the formula

$$A = L^{-x/\nu} F[(p-p_c)L^{1/\nu}] \tag{5}$$

where A is a quantity which becomes critical, $A \sim |p-p_c|^x$, in the asymptotic limit. A, for example, can be either χ or P. The function F is a suitable scaling function. At $p = p_c$ the quantity A varies as $L^{-x/\nu}$. This result can be used to determine the average cluster size exponent γ by calculating A for various values of L and using the value of ν obtained from equation (1). The exponent β can also be determined in the same manner. For a particular lattice size L and for $p = \langle p_c(L)\rangle$, let the size of the largest cluster which spans the lattice be S_∞. The spanning cluster is a fractal with fractal dimension D defined by

$$S_\infty \sim L^D. \tag{6}$$

The fractal dimension D can again be written as

$$D = d - \beta/\nu \tag{7}$$

since $P = S_\infty/L^d$ in (4). From equation (6) the fractal dimension D can be determined and from equation (7), knowing D, ν, the value of β can be obtained.

From scaling theory which is supposed to be valid in the critical region, n_s, the number of clusters of size s per site of the lattice is assumed to scale as (Stauffer 1985)

$$n_s = s^{-\tau} f[(p-p_c)s^{\sigma}] \tag{8}$$

for large clusters near the percolation threshold. τ and σ are exponents to be determined numerically. All the known percolation exponents are expressed in terms of these two exponents τ and σ. For example,

$$\gamma = (3-\tau)/\sigma \qquad \text{and} \qquad \beta = (\tau - 2)/\sigma. \tag{9}$$

A verification of the scaling function form in (8) is possible by plotting $n_s(p)/n_s(p_c)$ against $(p-p_c)s^{\sigma}$. If the scaling form is true, then for sufficiently large clusters and for different values of p, the data should collapse on to a single curve. Once γ and β are known through finite-size scaling analysis of data obtained, the exponent σ can be calculated from (9) and a test of the scaling form as given in (8) can be undertaken. In the next section, we give the values of $p_c(\infty)$ and the different critical exponents obtained by us using the method and formulae mentioned in this section. We also give the result of the test of scaling theory.

3. Results

The average value of the percolation threshold $\langle p_c(L)\rangle$ and the spread $\Delta(L)$ in the estimates have been obtained for lattice size $L \times L$ with L ranging from 50–140 in steps of 10. The data are listed in table 1. In figure 2 we have plotted $-\log \Delta(L)$ against

Table 1. Data for $\langle p_c(L)\rangle$, $D(L)$ and $\langle S_\infty\rangle$ for all $L \times L$ lattices and over N realizations.

L	N	$\langle p_c(L)\rangle$	$\Delta(L)$	$\langle S_\infty\rangle$
50	7000	0.7039	0.0210	713.398
60	5000	0.7053	0.0193	994.024
70	4500	0.7067	0.0180	1325.155
80	4000	0.7074	0.0170	1745.984
90	3500	0.7083	0.0165	2146.475
100	3000	0.7083	0.0157	2641.420
110	3000	0.7093	0.0151	3199.691
120	3000	0.7093	0.0147	3731.463
130	3000	0.7101	0.0143	4452.223
140	3000	0.7107	0.0139	5036.832

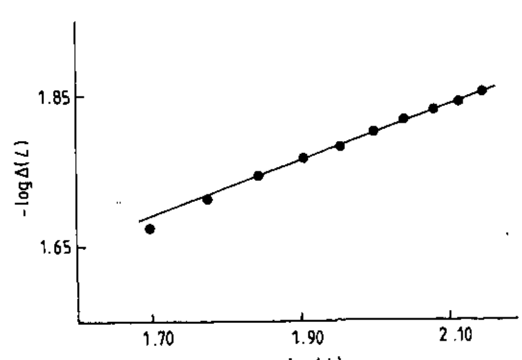

Figure 2. A plot of $-\log \Delta(L)$ against $\log(L)$ for a square lattice of size $L \times L$. The slope of the straight line gives $1/\nu = 0.396$.

log(L) which is a straight line in agreement with equation (1). The slope of the straight line gives the reciprocal of the correlation length exponent ν, $1/\nu = 0.396$. For undirected percolation, $1/\nu = 0.75$. In figure 3, we have plotted $\langle p_c(L)\rangle$ against $L^{-1/\nu}$ using the value obtained for ν. The plot is approximately a straight line and from equation (2) we obtained the value of $p_c(\infty) = 0.723 \pm 0.001$. The value of ν obtained is 2.53 ± 0.02. The errors in the values of $p_c(\infty)$ and ν have been obtained following the procedure outlined by Levinshtein *et al* (1976). Figure 4 shows a plot of the logarithm of the average spanning cluster size at the percolation threshold $p_c = \langle p_c(L)\rangle$ against log(L). The slope of the straight line gives the fractal dimension D. The value of D obtained is $D = 1.956 \pm 0.009$. Using equation (7) and putting the Euclidean dimension $d = 2$,

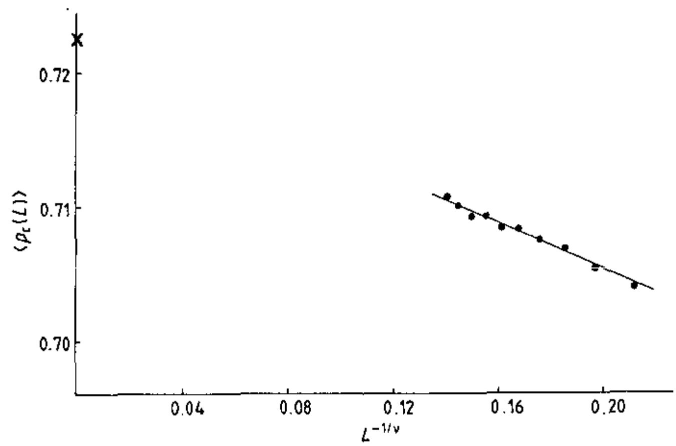

Figure 3. A plot of $\langle p_c(L)\rangle$ against $L^{-1/\nu}$. The intercept on the vertical axis gives $p_c(\infty) = 0.723 \pm 0.001$.

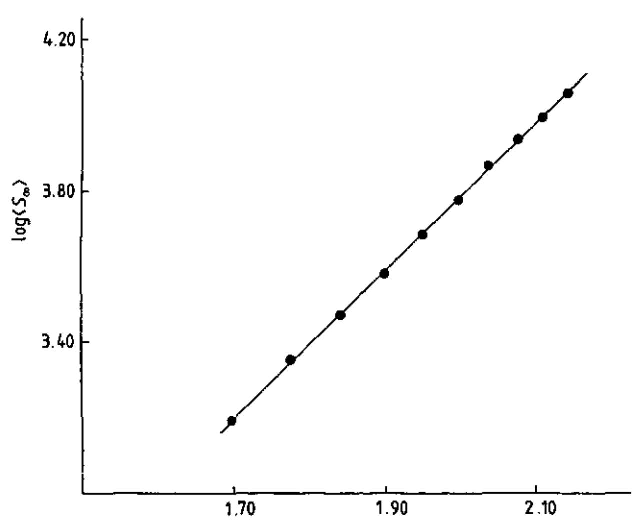

Figure 4. A plot of $\log\langle S_\infty\rangle$ against log(L) at $p = \langle p_c(L)\rangle$ where $\langle S_\infty\rangle$ is the average spanning cluster size. The slope of the straight line gives the fractal dimension D of the spanning cluster, $D = 1.956 \pm 0.009$.

the ratio of exponents β/ν is obtained as $\beta/\nu = 0.044 \pm 0.009$. In figure 5 we have plotted the logarithm of the average cluster size excluding the spanning cluster (equation (3)) against $\log(L)$ at the percolation threshold. The plot obtained is a straight line. According to the finite-size scaling formula (equation (5)), the slope of the straight line gives the ratio of exponents γ/ν, $\gamma/\nu = 2.055 \pm 0.025$. The errors quoted for D and γ/ν are the standard least-squares fit errors taking into account the statistical error of each single data point. A direct estimate of the exponent τ can be obtained in the following manner. Instead of using the scaling form given in equation (8), we use a modified scaling expression (Dhar and Barma 1981). Let $P_s(p)$ be the probability that a cluster of s sites is obtained ($P_s(p) \propto sn_s(p)$). If $F_s(p)$ is the probability that the number of sites in a cluster is strictly greater than s, then

$$F_s(p) = 1 - \sum_{i=1}^{s} P_i(p). \tag{10}$$

In the critical region $p \to p_c$, the probabilities $P_s(p)$ are expected to obey the scaling behaviour

$$P_s(p) \sim s^{-\tau+1} f[(p-p_c)s^{\sigma}] \tag{11}$$

where $f(x)$ is the scaling function. From equations (10) and (11), $F_s(p)$ has the scaling form

$$F_s(p) \sim s^{-\tau+2} h[(p-p_c)s^{\sigma}] \tag{12}$$

where the function $h(x)$ may be obtained from $f(x)$ by quadrature. This particular analysis is carried out for only one lattice size, in our case the size is 140×140. In figure 6, $-\ln F_s(p)$ has been plotted against $\ln(s)$ at the percolation threshold. The plot is a straight line from the slope of which $\tau - 2$, hence τ is obtained. The value of τ is $\tau = 2.02 \pm 0.01$. The error in the value of τ is determined using the maximum likelihood method (Dhar and Barma 1981). From equation (9), knowing any two of the exponents τ, γ and β, the exponent σ can be calculated. The value of σ obtained

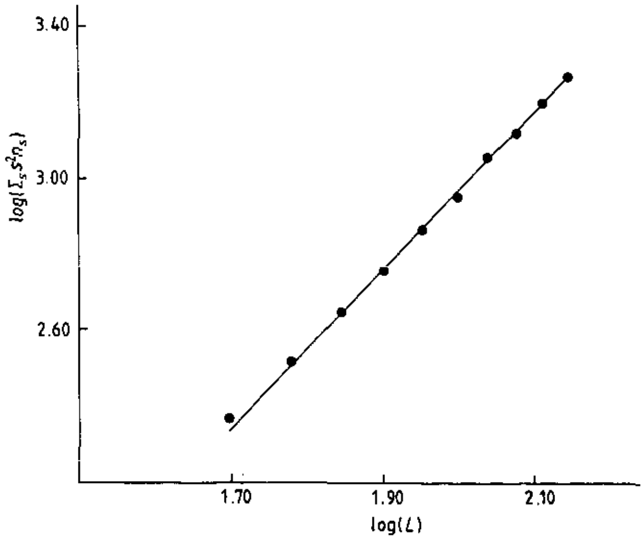

Figure 5. A plot of $\log(\Sigma_s s^2 n_s)$ against $\log(L)$ at $p = \langle p_c(L) \rangle$. The slope gives $\gamma/\nu = 2.055 \pm 0.025$.

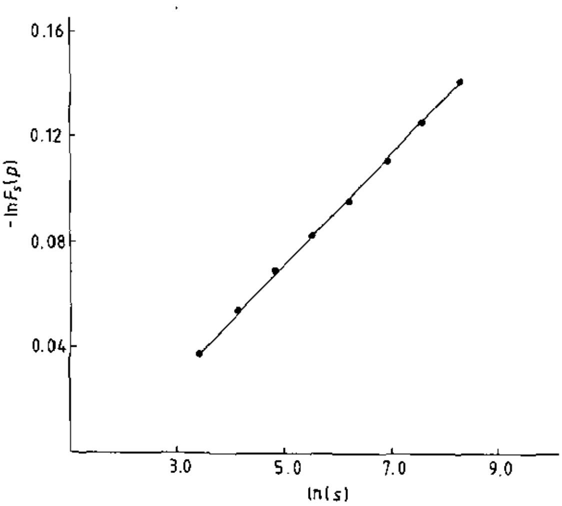

Figure 6. A plot of $-\ln F_s(p)$ against $\ln(s)$ for $p = \langle p_c(L) \rangle$. The slope gives $\tau - 2 = 0.02 \pm 0.01$.

is $\sigma = 0.19 \pm 0.02$. The test of the scaling form given in equation (8) has been done for a lattice size 140×140 with p_c equal to $\langle p_c(L) \rangle$. For $n_s(p)$, $n_s(p_c)$, the number of clusters per site of the lattice in a particular bin has been used, the size s being the geometric mean of the bounding cluster sizes of the bin. Figure 7 shows a plot of $n_s(p)/n_s(p_c)$

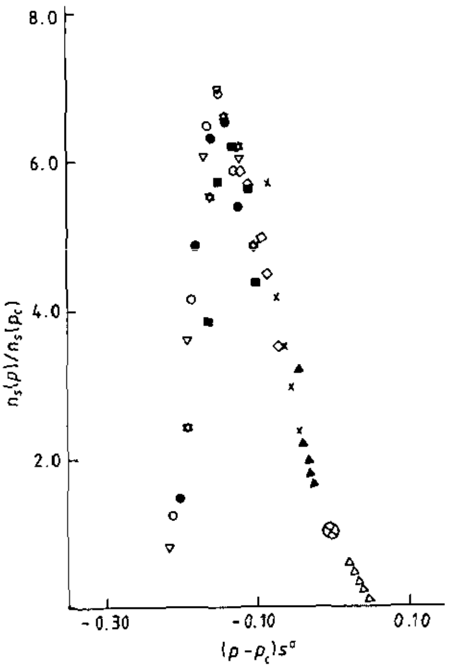

Figure 7. A plot of $n_s(p)/n_s(p_s)$ against $(p-p_c)s^\sigma$ for $\sigma = 0.19$. Data for $p - p_c = 0.01$ (open triangle), -0.01 (full triangle), -0.03 (open diamond), -0.04 (full rectangle), -0.045 (star), -0.05 (full circle), -0.525 (open circle), -0.055 (inverted open triangle) fall roughly on to a single curve in agreement with scaling theory predictions. The circled cross denotes the location of the point (0, 1).

against the scaling variable $(p-p_c)s^{\sigma}$ for nine different values of p. The data for different values of p have been marked by different symbols, $|p-p_c|$ being in the range 0.01–0.055. The circled cross denotes the location of the point (0, 1). Finally in table 2 we list the values of all the exponents we have obtained through finite-size scaling analysis. For the sake of comparison we also list the values of the exponents for undirected percolation.

Table 2. Numerical values of the critical exponents $\nu, \beta, \gamma, \tau, \sigma$ using finite-size scaling theory. The table also gives the values of the exponents for undirected percolation. The rational numbers are (presumably) exact results (Stauffer 1985).

Percolation model	ν	β	γ	τ	σ
Spiral	2.53 ±0.02	0.11 ±0.01	5.19 ±0.03	2.02 ±0.01	0.19 ±0.02
Undirected	4/3	5/36	43/18	187/91	36/91

4. Discussion

The values of the critical exponents obtained and listed in table 2 are widely different from the corresponding values in the cases of undirected and directed percolation. One can conclude that spiral or rotationally constrained percolation belongs to a different universality class. One interesting feature of spiral percolation is that the spanning cluster is nearly compact, the fractal dimension D being 1.956 ± 0.009 whereas the spanning cluster in other types of percolation is highly fragmented with the fractal dimension significantly lower than the Euclidean dimension. The fact that rotational constraint gives rise to a robust, compact structure has already been noted (Bose and Ray 1987) in a study on spiral lattice site animals. The rotational constraint has the effect equivalent to that of a centripetal force inhibiting fragmented outward growth and giving rise to a drawn-in, nearly compact structure. The use of finite-size scaling theory is justified by good agreement of data obtained numerically with scaling theory formulae (figures 2–6). Evidence for a scaling form of the cluster distribution function is obtained in figure 7 where data for different p have collapsed, though not very sharply, onto a single curve. A cleaner collapse is expected only for a larger lattice size. Figure 7 clearly shows that the maximum is located below the percolation threshold. This is also true in the case of undirected percolation (Stauffer 1985); a symmetric scaling function is obtained only in the case of percolation on a Bethe lattice. The value of the exponent τ listed in table 2 has been obtained from equation (12). If the values of β, γ shown in table 2 are substituted in equation (9), the value of τ comes out to be $\tau = 2.0201$ which agrees closely with the value of τ quoted in table 2. This close agreement is a proof of the validity of the scaling equations given in (9). Spiral percolation is similar to directed percolation since for both the models a directional constraint is operative. In the first case the directional constraint is rotational in nature and in the second case percolation occurs only in certain specific directions. The critical exponents for the two models have, however, different values. Different directional constraints thus lead to different universality classes. As mentioned in section 1, Ray and Bose (1988) have studied spiral percolation on an undirected percolation spanning cluster for lattice size up to 60×60. The value of the correlation length exponent ν_s obtained is $\nu_s = 1.404 \pm 0.012$. This value differs considerably from the value of ν obtained by us for spiral percolation on the square lattice. In the previous model both the maximum lattice size and the number of Monte Carlo realizations over which averages are taken are smaller than those in the present simulation. The common lattice sizes used in the two studies are $L = 50$ and 60. The data for these two sizes indicate a significant difference in the values of ν_s and ν. Thus the large discrepancy in the values of the correlation length exponent is not due to significant correction to scaling as a function of lattice size but rather leads to the conjecture that the two models belong to different universality classes. Spiral percolation is expected to occur in disordered systems when a rotational force field is present. An example of rotational constraint is the motion of a charged particle in a plane in the presence of a magnetic field perpendicular to the plane. A cycloidal trajectory as in the presence of crossed electric and magnetic fields is an example of a path traced out under rotational constraint. Studies along these lines are in progress and the results will be reported elsewhere.

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